

Paintings on a Wall

In order to optimize the viewing space for its patrons, a museum has placed size restrictions on rectangular paintings that will be hung on a particular wall.

The perimeter of a painting must be between 64 inches and 100 inches, inclusive. The area of the painting must be between 200 square inches and 500 square inches.

1. You are in charge of determining possible perimeter and area combinations for paintings to be hung on the wall. Write inequalities to describe the perimeter and area restrictions in terms of the length and the width of the rectangles. Graph the resulting system.
2. Algebraically and with technology, determine the vertices of the region defined by the inequalities in problem 1.
3. Describe the location of the points on your graph where the dimensions of the painting result in each of the following:
 - a. The perimeter and area are acceptable.
 - b. The perimeter is too short or too long, but the area is acceptable.
 - c. The perimeter is acceptable, but the area is too small or too large.
 - d. Neither the perimeter nor the area is acceptable.

Explain how you arrived at your responses.



Notes

Materials:

Graphing calculator

Algebra II TEKS Focus: (2A.3) Foundations for functions. The student formulates systems of equations and inequalities from problem situations, uses a variety of methods to solve them, and analyzes the solutions in terms of the situations.

The student is expected to:

(A) analyze situations and formulate systems of equations in two or more unknowns or inequalities in two unknowns to solve problems.

(B) use algebraic methods, graphs, tables, or matrices, to solve systems of equations or inequalities.

(C) interpret and determine the reasonableness of solutions to systems of equations or inequalities for given contexts.

Additional Algebra II TEKS: (2A.1) Foundations for functions. The student uses properties and attributes of functions and applies functions to problem situations.

Scaffolding Questions:

- What information is known?
- What are you expected to show?
- What compound inequality can you write to describe the restrictions on the perimeter?
- What compound inequality can you write to describe the restrictions on the area?
- How can you use the inequalities to write functions that describe the boundaries of the region containing acceptable values for width and length of a painting?
- Describe the process you might use to graph the compound inequality.
- What functions must be graphed to model a combined inequality such as $2 \leq x + y \leq 5$?
- What kind of systems must you solve to get the vertices of the boundary?
- What algebraic method will you use to solve each system? Why?
- Point to each region in the plane determined by the boundary functions and describe whether the perimeter and area meet the requirements in that region.

Sample Solutions:

1. Let l = the length in inches of a painting and w = the width in inches of the painting.

Since the perimeter of the painting must be between 64 inches and 100 inches, we know that

$$64 \leq 2(l + w) \leq 100$$

which gives

$$32 \leq l + w \leq 50$$

$$32 - w \leq l \leq 50 - w$$

Since the area of the painting must be between 200 square inches and 500 square inches, we have

$$200 \leq lw \leq 500$$

which gives

$$\frac{200}{w} \leq l \leq \frac{500}{w}$$

On the graphing calculator, the length will be represented by Y and the width will be represented by x .

Function Rule:

$$l = 32 - w$$

$$l = 50 - w$$

$$l = \frac{200}{w}$$

$$l = \frac{500}{w}$$

Calculator Rule:

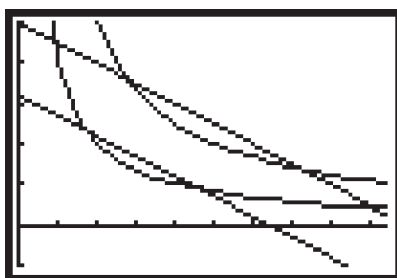
$$Y_1 = 32 - x$$

$$Y_2 = 50 - x$$

$$Y_3 = \frac{200}{x}$$

$$Y_4 = \frac{500}{x}$$

Here is the calculator graph of the system with window settings $0 \leq x \leq 47$, $-10 \leq y \leq 50$:



The region that is the solution set to the system of inequalities is the closed region between the two lines and the two curves.

- We use the intersect feature of the calculator to determine the coordinates of the vertices. Starting with the upper left vertex and moving counterclockwise around the region, they are:

The intersection of Y_2 and Y_3 : A(4.38, 45.62)

The intersection of Y_1 and Y_3 : B(8.52, 23.48)

The student is expected to:

(A) identify the mathematical domains and ranges of functions and determine reasonable domain and range values for continuous and discrete situations.

(B) collect and organize data, make and interpret scatterplots, fit the graph of a function to the data, interpret the results, and proceed to model, predict, and make decisions and critical judgments.

Connection to TAKS:

Objective 1: The student will describe functional relationships in a variety of ways.

Objective 3: The student will demonstrate an understanding of linear functions.

Objective 4: The student will formulate and use linear equations and inequalities.

Objective 7: The student will demonstrate an understanding of two- and three-dimensional representations of geometric relationships and shapes.

Objective 10: The student will demonstrate an understanding of the mathematical processes and tools used in problem solving.

- The intersection of Y_1 and Y_3 : C(23.48, 8.52)
- The intersection of Y_2 and Y_3 : D(45.62, 4.38)
- The intersection of Y_2 and Y_4 : E(36.18, 13.82)
- The intersection of Y_2 and Y_4 : F(13.82, 36.18)

To find the vertices algebraically, we solve the following systems by substitution:

$$\begin{aligned}
 Y_1 &= 32 - x \\
 Y_3 &= \frac{200}{x} \\
 \text{Let } Y_1 &= Y_3 \\
 32 - x &= \frac{200}{x} \\
 32x - x^2 &= 200 \\
 x^2 - 32x + 200 &= 0 \\
 x &= \frac{32 \pm \sqrt{(-32)^2 - 4 \cdot 1 \cdot 200}}{2} \\
 x &= 23.48 \text{ or } x = 8.52
 \end{aligned}$$

This gives the x-coordinate for vertices B and C.

Substitute for x in Y_1 to get the y-coordinates.

$$\begin{aligned}
 Y_2 &= 50 - x \\
 Y_3 &= \frac{200}{x} \\
 \text{Let } Y_2 &= Y_3 \\
 50 - x &= \frac{200}{x} \\
 50x - x^2 &= 200 \\
 x^2 - 50x + 200 &= 0 \\
 x &= \frac{50 \pm \sqrt{(-50)^2 - 4 \cdot 1 \cdot 200}}{2} \\
 x &= 45.62 \text{ or } x = 4.38
 \end{aligned}$$

This gives the x-coordinate for vertices D and A.

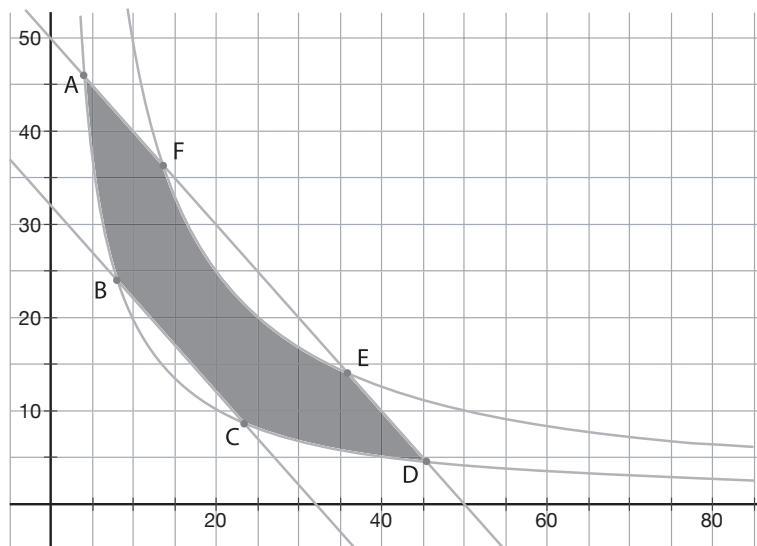
Substitute for x in Y_2 to get the y-coordinates.

$$\begin{aligned}
 Y_2 &= 50 - x \\
 Y_4 &= \frac{500}{x} \\
 \text{Let } Y_2 &= Y_4 \\
 50 - x &= \frac{500}{x} \\
 50x - x^2 &= 500 \\
 x^2 - 50x + 500 &= 0 \\
 x &= \frac{50 \pm \sqrt{(-50)^2 - 4 \cdot 1 \cdot 500}}{2} \\
 x &= 36.18 \text{ or } x = 13.82
 \end{aligned}$$

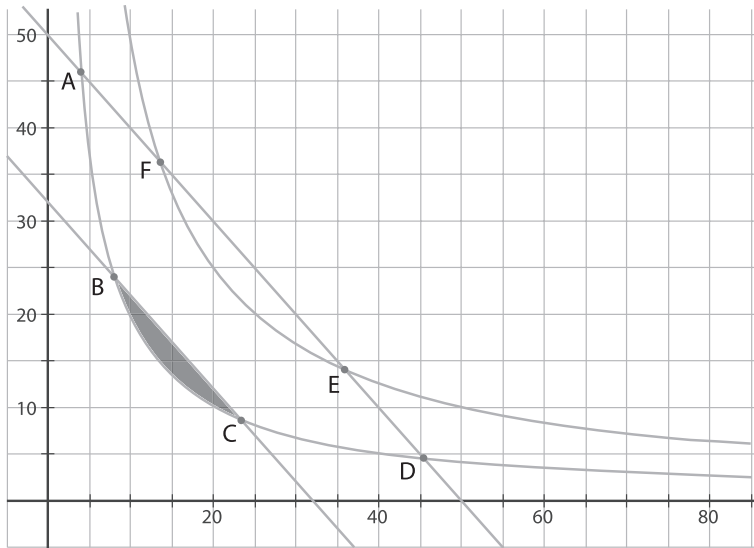
This gives the x -coordinate for vertices E and F.

Substitute for x in Y_2 to get the y -coordinates.

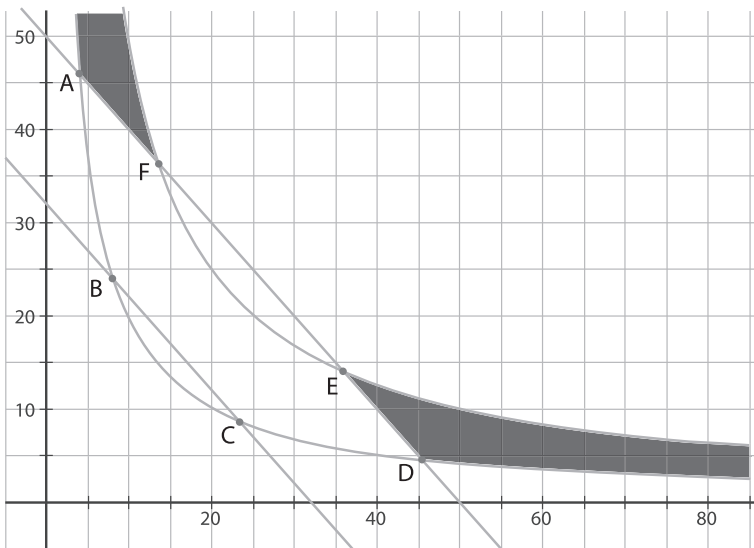
3. Remember that the coordinates (x, y) of the points in the plane represent (width, length) of the painting. Therefore, we can consider points only in the first quadrant.
 - a. The region where the perimeter and area are acceptable is the closed region between the lines and the curves.



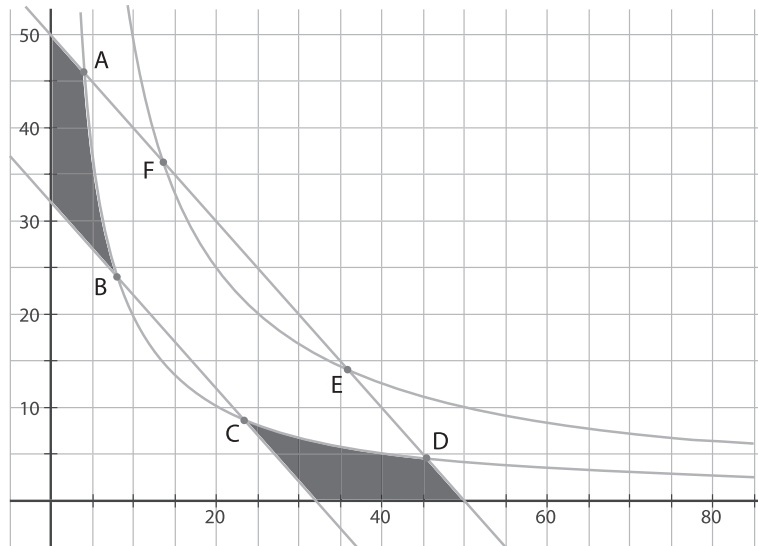
- b. For the perimeter to be too short and the area acceptable, we need first quadrant points below the line $y = 32 - x$ but between or on the curves.



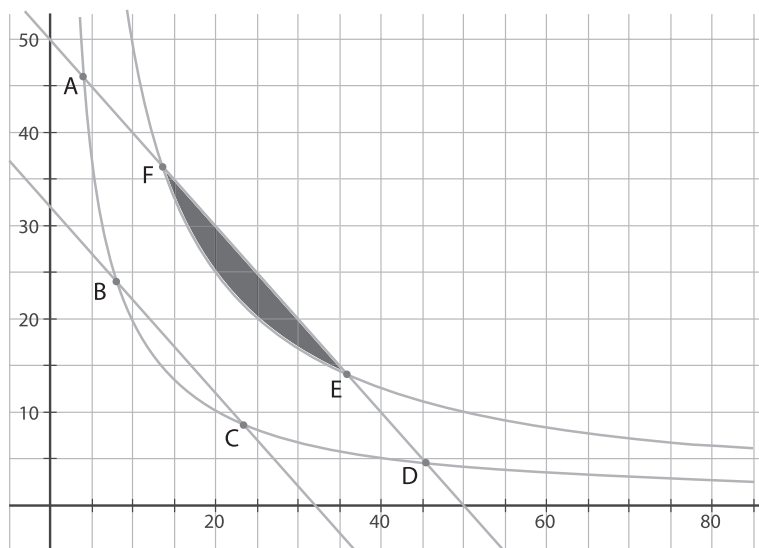
For the the perimeter to be too long and the area acceptable, we need first quadrant points above the line $y = 50 - x$ but between or on the curves.



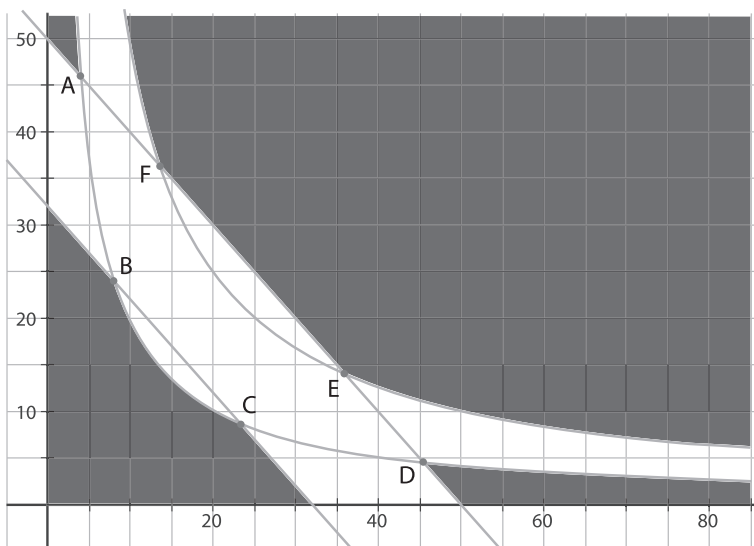
- c. For the perimeter to be acceptable and the area too small, we need first quadrant points between or on the lines but below the curve $y = \frac{200}{x}$.



For the perimeter to be acceptable and the area too large, we need first quadrant points between or on the lines but above the curve $y = \frac{500}{x}$.



- d. If the perimeter and area are both unacceptable, we need first quadrant points both outside the lines and the curves.



Extension Questions:

- How is the region representing acceptable dimensions for a painting different from regions you encountered in linear programming problems?

The boundaries are not all defined by linear functions. Three boundaries are segments, defined by the two linear (perimeter) functions. Three boundaries are defined by the two reciprocal (area) functions.

- Which vertex or vertices minimize perimeter and area, subject to the restrictions? How do you know?

To minimize both perimeter and area you must be at the intersection of

$Y_1 = 32 - x$ and $Y_3 = \frac{200}{x}$ because these give the lower limits on perimeter and area. This is vertex B or C. The painting can have dimensions 8.52 inches by 23.48 inches. To maximize both perimeter and area, you must be at the intersection of

$Y_2 = 50 - x$ and $Y_4 = \frac{500}{x}$ because these give the upper limits on perimeter and area. This is vertex E or F. The painting can have dimensions 36.18 inches by 13.82 inches.

- Does the graph have any symmetry?

Yes. It is symmetric with respect to the line $y = x$.

- Suppose the area restriction were replaced with the restriction that the diagonal of a painting must be between 25 inches and 40 inches in length. How will this change your responses to the previous two questions?

The restrictions on the perimeter stay the same, but we must replace area restrictions with diagonal restrictions.

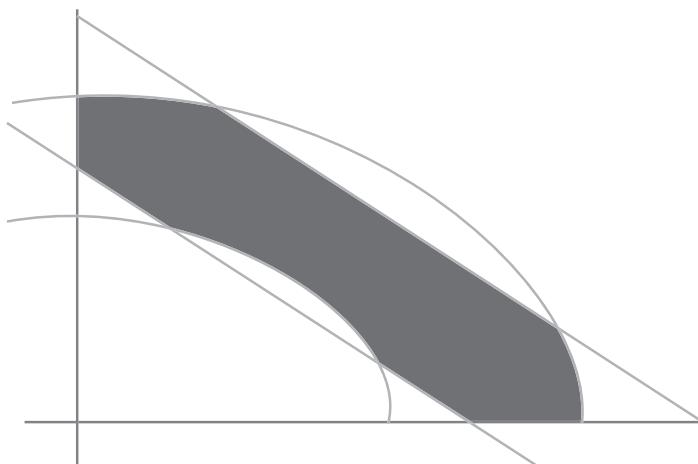
Let d = the length of the painting's diagonal in inches. Then, since $d^2 = l^2 + w^2$,

$$\begin{aligned}
 25 \leq d \leq 40 &\Rightarrow 25^2 \leq d^2 \leq 40^2 \\
 &25^2 \leq l^2 + w^2 \leq 40^2 \\
 &25^2 - w^2 \leq l^2 \leq 40^2 - w^2 \\
 &\sqrt{25^2 - w^2} \leq l \leq \sqrt{40^2 - w^2}
 \end{aligned}$$

The area boundary equations are replaced with

$$l = \sqrt{25^2 - w^2} \text{ and } l = \sqrt{40^2 - w^2}$$

The graph of the region representing acceptable dimensions is the first quadrant region between the lines and between the semi-circles. It does not include points on the axes, since that would give you zero width or zero length.



To find where the perimeter lines and diagonal circles intersect, we solve the systems consisting of Y_1 and Y_3 and of Y_2 and Y_4 . For example,

$$\begin{aligned}
 Y_1 &= 32 - x \\
 Y_3 &= \sqrt{25^2 - x^2} \\
 \text{Let } Y_1 &= Y_3 \\
 32 - x &= \sqrt{25^2 - x^2} \\
 x^2 - 64x + 32^2 &= 25^2 - x^2 \\
 2x^2 - 64x + 32^2 - 25^2 &= 0
 \end{aligned}$$

Apply the quadratic formula to get $x = 8.48$ inches or $x = 23.52$ inches.

Similarly, we solve for the intersection of Y_2 and Y_4 to get $x = 11.77$ inches or $x = 38.23$ inches.

The vertex points are $(8.48, 23.52)$, $(23.52, 8.48)$, $(11.77, 38.23)$ and $(38.23, 11.77)$.